On Cooperative Games Arising from Deterministic Auction Situations

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Summary

Scenario

Auction games and ring games

The core

The Shapley value

Further developments

Scenario

Auction: a powerful and efficient way for selling an object when the owner has limited info on its value

Examples of objects: paintings, flowers, vines, bonds, etc.

In an auction there is one *seller* and several *bidders*; there can be a *single* object or *multiple* objects; the seller may fix a *reservation price*

Types of auctions: Open ascending price (English auction)
Open descending price (Dutch auction)
Sealed-bid first-price
Sealed-bid second-price (Vickrey auction)

Here
Single object, sealed-bid second-price auctions with complete information

Auction games and ring games

$N = \{1, ..., n\}$	set of bidders
$n + 1$	seller
$w \in \mathbb{R}^n$	vector of the evaluations of the bidders
	bidder $i \in N$ evaluates the object w_i; $w_i \geq w_{i+1}$
w_{n+1}	reservation price (evaluation) of the seller; $w_{n+1} \leq w_i, \forall\ i \in N$

Notation for *1-object auction situation*

$$< w_1, w_2, ..., w_n; w_{n+1} : 1 >$$

Definition 1 *Let* $< w_1, w_2, ..., w_n; w_{n+1} : 1 >$ *be a 1-object auction situation; the corresponding* auction game *is* $(N \cup \{n+1\}, a)$
where $N \cup \{n+1\}$ *is set of players*
a *is the characteristic function defined by*

$$a(S) = \begin{cases} \max_{i \in S \setminus \{n+1\}} \{w_i - w_{n+1}\} & \text{if } n+1 \in S \\ 0 & \text{if } n+1 \notin S \end{cases} \qquad \forall\ S \subseteq N \cup \{n+1\}$$

This definition does not depend on the auction type (market situation)

$(N \cup \{n+1\}, a)$ is a total big boss game, with $n+1$ as big boss (see Muto, Nakayama, Potters and Tijs, 1988)

The agents involved in an auction may collude (illegally) in order to increase their gains, i.e. they form a *ring*

In particular:

- The bidders may collude against the seller $\longrightarrow$ *buyer ring game*
- The seller and one or more bidders may collude against the first bidder $\longrightarrow$ *mixed ring game*

Definition 2 *Let* $< w_1, w_2, ..., w_n; w_{n+1} : 1 >$ *be a 1-object auction situation; the corresponding buyer ring game is* (N, r)
where N *is set of players*
r *is the characteristic function defined by*

$$r(S) = \begin{cases} w_1 - w_{k+1} & \text{if } [1,k] \subset S, [1,k+1] \not\subset S, k \in N \\ 0 & \text{if } 1 \notin S \end{cases} \qquad \forall\ S \subseteq N$$

with $[1, k] = \{1, ..., k\}$

(N, r) is a peer group game (see Branzei, Fragnelli and Tijs, 2002)

Definition 3 *Let* $< w_1, w_2, ..., w_n; w_{n+1} : 1 >$ *be a 1-object auction situation; the corresponding mixed ring game is* $(N \setminus \{1\} \cup \{n+1\}, r^1)$
where $N \setminus \{1\} \cup \{n+1\}$ *is set of players*
r *is the characteristic function defined by*

$$r(S) = \begin{cases} w_1 - w_2 & \text{if } n+1 \in S, |S| \geq 2 \\ 0 & \text{if } n+1 \notin S \text{ or } S = \{n+1\} \end{cases} \qquad \forall\ S \subseteq N \setminus \{1\} \cup \{n+1\}$$

Example 1 *Consider the following 1-object auction situation* $< 210, 90, 60; 0 : 1 >$

- $< \{1, 2, 3, 4\}, a >$ *is given by:*

$$a(14) = a(124) = a(134) = a(1234) = 210$$
$$a(24) = a(234) = 90$$
$$a(34) = 60$$
$$a(S) = 0 \textit{ for each other different coalition } S$$

- $< \{1, 2, 3\}, r >$ *is given by:*

$$r(123) = 210$$
$$r(12) = 150$$
$$r(1) = r(13) = 120$$
$$r(S) = 0 \textit{ for each other different coalition } S$$

- $< \{2, 3, 4\}, r^1 >$ *is given by:*

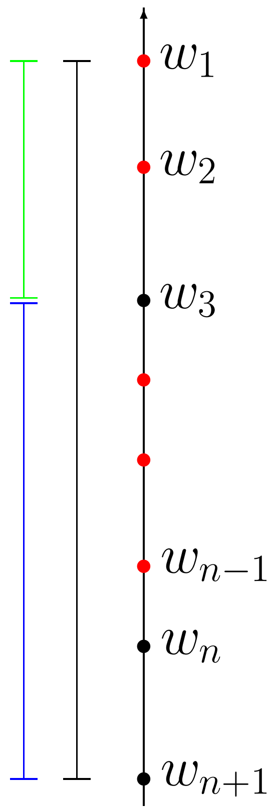
$$r^1(24) = r^1(34) = r^1(234) = 120$$
$$r^1(S) = 0 \textit{ for each other different coalition } S$$

Definition 4 *Given a game (N, v) the corresponding* dual game (N, v^*) *has the same set of players N and the characteristic function is defined by:*

$$v^*(S) = v(N) - v(N \setminus S), \forall\ S \subseteq N$$

Given a 1-object auction situation $< w_1, w_2, ..., w_n; w_{n+1} : 1 >$, the dual game of the auction game, restricted to N, i.e. considering only the coalitions that do not include player $n+1$, coincides with the buyer ring game:

$$a^*(S) = a(N \cup \{n+1\}) - a(N \cup \{n+1\} \setminus S) = r(S), \forall\ S \subseteq N$$



The Core

Given a game (N, v) the core is the set

$$Core(v) = \left\{ (x_i)_{i \in N} \in \mathbb{R}^{|N|} \ s.t. \ \sum_{i \in S} x_i \geq v(S), \forall \ S \subset N \ and \ \sum_{i \in N} x_i = v(N) \right\}$$

If an allocation $x \notin Core(v)$ is proposed, then it is not possible to be sure that no player recedes from the grand coalition N

Auction games and ring games have non empty core

Theorem 1 *Let* $< w_1, w_2, ..., w_n; w_{n+1} : 1 >$ *be a 1-object auction situation. Then:*

(i) $Core(a) = conv\,\{(0, 0, ..., 0, w_1 - w_{n+1}), (w_1 - w_2, 0, ..., 0, w_2 - w_{n+1})\}$

(ii) $$Core(r) = \left\{ x \in \mathbb{R}^n \Bigg/ \begin{array}{c} \sum_{i \in N} x_i = w_1 - w_{n+1}, \\ w_1 - w_2 \leq x_1 \leq w_1 - w_{n+1}, \\ 0 \leq x_i \leq w_i - w_{n+1}, i \in N \setminus \{1\} \end{array} \right\}$$

(iii) $$Core(r^1) = \begin{cases} conv\,\{(w_1 - w_2, 0), (0, w_1 - w_2)\} & \text{if } n = 2 \\ \{(0, ..., w_1 - w_2)\} & \text{if } n \geq 3 \end{cases}$$

The minimum benefit for each bidder in the buyer ring game coincides with his maximum in the auction game; moreover the grand coalition of the bidders takes all

Example 2 *Consider the situation* $< 210, 90, 60; 0 : 1 >$ *of Example 1*

(i) $Core(a) = conv\,\{(0, 0, 0, 210), (120, 0, 0, 90)\}$

(ii) $Core(r) = conv\,\{(210, 0, 0), (150, 0, 60), (120, 90, 0), (120, 30, 60)\}$

(iii) $Core(r^1) = \{(0, 0, 120)\}$

The Shapley value

The Shapley value (Shapley, 1953) is a classical game theoretical solution; it assigns to each player his average marginal contribution w.r.t. all the possible permutations of the players:

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi} \left[v(P(\pi; i) \cup \{i\}) - v(P(\pi; i)) \right]$$

where $P(\pi; i)$ is the set of the predecessors of i in permutation π

Auction games and ring games have very simple formulas for computing Shapley value

Theorem 2 *Let* $< w_1, w_2, ..., w_n; w_{n+1} : 1 >$ *be a 1-object auction situation. Then:*

(i) $$\phi_i(a) = \begin{cases} \sum_{k=i,\ldots,n} \frac{1}{k(k+1)}(w_k - w_{k+1}) & \textit{if } i \in N \\ \sum_{k=i,\ldots,n} \frac{1}{k(k+1)} w_k - \frac{n}{n+1} w_{n+1} & \textit{if } i = n+1 \end{cases}$$

Difference of an airport game (Littlechild and Owen, 1973) and a generalized airport game (Norde, Fragnelli, García-Jurado, Patrone and Tijs, 2002)

(ii) $$\phi_i(r) = \sum_{k=i,\ldots,n} \frac{1}{k}(w_k - w_{k+1}), \forall\ i \in N$$

Peer group game (Branzei, Fragnelli and Tijs, 2002)

(iii) $$\phi_i(r^1) = \begin{cases} \frac{1}{n(n-1)}(w_1 - w_2) & \textit{if } i \in N \setminus \{1\} \\ \frac{n-1}{n}(w_1 - w_2) & \textit{if } i = n+1 \end{cases}$$

By symmetry of players in $N \setminus \{1\}$ *and efficiency*

Example 3 *Consider the situation* $< 210, 90, 60; 0 : 1 >$ *of Example 1*

(i) $\phi_i(a) = (70, 10, 5, 125)$

(ii) $\phi_i(r) = (155, 35, 20)$

(iii) $\phi_i(r^1) = (20, 20, 80)$

Further developments

PMAS

Bi-MAS

Multiobject auctions

Relations between solutions for the three games

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Thanks!